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The Geometry of D=11 Killing Spinors

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ABSTRACT

We propose a way to classify all supersymmetric configurations of D=11 supergravity using the G-structures defined by the Killing spinors. We show that the most general bosonic geometries admitting a Killing spinor have at least an $SU(5)$ or an $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, depending on whether the Killing vector constructed from the Killing spinor is time-like or null, respectively. In the former case we determine what kind of $SU(5)$ structure is present and show that almost all of the form of the geometry is determined by the structure. We also deduce what further conditions must be imposed in order that the equations of motion are satisfied. We illustrate the formalism with some known solutions and also present some new solutions including a rotating generalisation of the resolved membrane solutions and generalisations of the recently constructed D=11 Gödel solution.

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1 Introduction

Supersymmetric solutions of supergravity theories have played a prominent role in many developments in string theory and it would be useful to have a systematic classification of all such solutions. When the fluxes are all set to zero we know that the supersymmetric geometries must admit covariantly constant spinors and hence must admit metrics with special holonomy. There are many results in the literature concerning special cases when the fluxes are non-vanishing, but a global picture has been lacking[1]. Here we shall propose a way to classify all supersymmetric solutions of D=11 supergravity, building on the work of [2, 3, 4, 5, 6, 7] using G-structures[2]. Moreover, it will be clear how to extend the ideas to any supergravity theory. Indeed a complete analysis for $D = 5$ minimal supergravity has already been carried out in [7] (for earlier work on the simpler case of $N = 2$ supergravity in $D = 4$ using different techniques, see [10]).

We start in section 2 by deriving a number of necessary conditions for a bosonic geometry, consisting of a metric and a four-form, to admit Killing spinors. We first construct differential forms of rank 0,...,5 from bi-linears of the Killing spinors. Fierz identities then give a number of algebraic conditions that these forms must satisfy, while the Killing spinor equation gives a number of differential constraints. For example, the vector fields dual to the one-forms K constructed from the Killing spinors are always Killing. When one of the K is timelike, we show that some of the differential conditions are those of generalised calibrations [11, 12, 13] for membranes and also for fivebranes, with a small extension for the latter case. The same differential conditions hold when K is null or, when there is more than one Killing spinor, spacelike, which suggests an interesting extension of the notion of generalised calibration.

In section 3, we argue that the notion of G-structures is key to interpreting and organising the results of section 2. We begin by recalling the notion of G-structures and their classification and then discuss how they can provide the basis for a classification of all supersymmetric solutions. One result is that any supersymmetric solution, i.e. preserving at least one Killing spinor (1/32 supersymmetry), will either have an $SU(5)$ or an $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure. The two cases are distinguished by whether the Killing vector is time-like or null, respectively.

[1]Recently the classification of maximally supersymmetric solutions of D=10,11 supergravity was carried out in [1].

[2]For other work relating G-structures to supergravity solutions with non-vanishing fluxes, see [8, 9].

We carry out a detailed analysis for the time-like case. As noted the time-like vector is a Killing vector and hence the $SU(5)$ structure in eleven-dimensions turns out to be mostly specified by an $SU(5)$ structure in the ten-dimension base space orthogonal to the orbits of the Killing vector. The only restriction on the ten-dimensional $SU(5)$ structure is that the class $\mathcal{W}_5$, to be defined later, is exact and related to the norm of the Killing vector. We find the general form of the geometry admitting a timelike Killing spinor. We find that, much, but not all, of the form of the geometry is determined by the $SU(5)$ structure. In particular, there is a component of the four-form field strength which is undetermined, because it drops out of the Killing spinor equation. The necessary and sufficient form of the geometry is presented in (4.20), (4.21) and (4.22). By analysing integrability conditions for the Killing spinor equation, we also determine the extra constraints imposed on those geometries admitting Killing spinors in order that they solve the equations of motion. The extra constraints are presented in (4.29) and (4.30). Our results allow us to obtain some vanishing theorems for compactifications with flux (for other such theorems in D=10 supergravity with NS three-form flux only, and assuming a restricted class of configurations, see [14, 15, 6]).

We illustrate the formalism with some known solutions and also present some new solutions in section 4. In [16, 17] (see also [18, 19, 20]) it was shown that the membrane solution with a transverse manifold of $SU(4)$ holonomy can be resolved by switching on additional four-form flux via a harmonic four-form. Here we will show that one can extend these solutions to include rotation. In [7] a D=5 generalisation of the Gödel solution was constructed. It was shown that it can be uplifted to D=11 where it then preserves 5/8 supersymmetry. The topology of the space is $\mathbb{R}^{11}$ and there is a rotational one-form that lives in an $\mathbb{R}^4$ factor. We will show that there are further solutions with more complicated rotation one-forms.

Section 5 briefly concludes.

2 Killing spinors and differential forms

The bosonic fields of D=11 supergravity consist of a metric, g, and a three-form potential A with four-form field strength $F = dA$. The action for the bosonic fields is given by

$$S = \frac{1}{2\kappa^2} \int d^{11}x\sqrt{-g}R - \frac{1}{2}F \wedge *F - \frac{1}{6}C \wedge F \wedge F \tag{2.1}$$

where $F = dC$. The equations of motion are thus given by

$$\begin{aligned} R_{\mu\nu} - \frac{1}{12}(F_{\mu\sigma_1\sigma_2\sigma_3}F_{\nu}{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{12}g_{\mu\nu}F^2) &= 0 \\ d * F + \frac{1}{2}F \wedge F &= 0 \end{aligned} \tag{2.2}$$

We are interested in bosonic solutions to the equations of motion that preserve at least one supersymmetry i.e. solutions that admit at least one Killing spinor, ϵ, which solves

$$\nabla_\mu \epsilon + \frac{1}{288}[\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}]F_{\nu_1\nu_2\nu_3\nu_4}\epsilon = 0 \tag{2.3}$$

Note that due to the presence of the four-form the supercovariant derivative appearing in (2.3) takes values in the Clifford algebra and not just the spin subalgebra. Our conventions are outlined in the appendix.

Note that in M-theory, the field-equation for the four-form receives higher order gravitational corrections [21, 22]:

$$d * F + \frac{1}{2}F \wedge F = (2\pi)^4 X_8, \tag{2.4}$$

where

$$X_8 = \frac{1}{(2\pi)^4}\left(-\frac{1}{768}(trR^2)^2 + \frac{1}{192}trR^4\right) \tag{2.5}$$

and we have used units where the M-fivebrane has tension given by $T_6 = 1/(2\pi)^3$. Since most of our analysis concerns the Killing spinor equation (2.3) this correction will not play a large role in the following.

Consider a geometry that admits N Killing spinors ϵ^i, $i = 1, .., N$. We can define the following one, two and five-forms that are symmetric in i, j:

$$\begin{aligned} K^{ij}_\mu &= \bar{\epsilon}^i \Gamma_\mu \epsilon^j \\ \Omega^{ij}_{\mu_1\mu_2} &= \bar{\epsilon}^i \Gamma_{\mu_1\mu_2} \epsilon^j \\ \Sigma^{ij}_{\mu_1\mu_2\mu_3\mu_4\mu_5} &= \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \epsilon^j \end{aligned} \tag{2.6}$$

We can also define zero, three and four-forms which are anti-symmetric in i, j

$$\begin{aligned} X^{ij} &= \bar{\epsilon}^i \epsilon^j \\ Y^{ij}_{\mu_1\mu_2\mu_3} &= \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\mu_3} \epsilon^j \\ Z^{ij}_{\mu_1\mu_2\mu_3\mu_4} &= \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\mu_3\mu_4} \epsilon^j \end{aligned} \tag{2.7}$$

2.1 Algebraic Relations

These differential forms are not all independent. They satisfy certain algebraic relations which are a consequence of the underlying Clifford algebra. One way of obtaining these is by repeated use of Fierz identities. Another approach will be mentioned later. Let us illustrate this by considering the case $i = j$ and dropping the ij indices, which covers the most general case when there is only one Killing spinor.

We first relate Ω^2 and Σ^2 to K^2. We use here the convention that for any p-form α we have:

$$\alpha^2 \equiv \frac{1}{p!}\alpha_{\mu_1\mu_2\dots\mu_p}\alpha^{\mu_1\mu_2\dots\mu_p}$$

By performing Fierz rearrangements on K^2, Ω^2 and Σ^2 in turn we find three linearly dependent equations. Solving them we find:

$$\begin{aligned}\Sigma^2 &= -6K^2 \\ \Omega^2 &= -5K^2\end{aligned} \tag{2.8}$$

We also find the following relations:

$$\Omega_{\mu_1}{}^{\sigma_1}\Omega_{\sigma_1}{}^{\nu_1} = -K_{\mu_1}K^{\nu_1} + \delta_{\mu_1}{}^{\nu_1}K^2 \tag{2.9}$$

$$\frac{1}{4!}\Sigma_{\mu_1}{}^{\sigma_1\sigma_2\sigma_3\sigma_4}\Sigma_{\sigma_1\sigma_2\sigma_3\sigma_4}{}^{\nu_1} \;=\; 14K_{\mu_1}K^{\nu_1} - 4\delta_{\mu_1}{}^{\nu_1}K^2 \tag{2.10}$$

$$i_K\Omega = 0 \tag{2.11}$$

$$i_K\Sigma = \frac{1}{2}\Omega\wedge\Omega \tag{2.12}$$

$$K^\sigma(*\Sigma)_{\sigma\nu_1\nu_2\nu_3\nu_4\nu_5} = \Omega_{\nu_1}{}^\sigma\Sigma_{\sigma\nu_2\nu_3\nu_4\nu_5} \tag{2.13}$$

$$\Omega\wedge\Sigma = \frac{1}{2K^2}K\wedge\Omega\wedge\Omega\wedge\Omega \tag{2.14}$$

These are by no means exhaustive.

2.2 Differential Relations

The covariant derivatives of the differential forms can be calculated by using the fact that a Killing spinor satisfies:

$$\overline{\nabla_\mu\epsilon^i} = \frac{1}{288}\bar{\epsilon}^i[\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} + 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}]F_{\nu_1\nu_2\nu_3\nu_4} \tag{2.15}$$

We find

$$
\begin{aligned}
\nabla_\mu K^{ij}_\nu &= \frac{1}{6}\Omega^{ij\sigma_1\sigma_2}F_{\sigma_1\sigma_2\mu\nu} + \frac{1}{6!}\Sigma^{ij\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} * F_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\mu\nu} \\
\nabla_\mu \Omega^{ij}_{\nu_1\nu_2} &= \frac{1}{3\cdot 4!}g_{\mu[\nu_1}\Sigma^{ij}{}_{\nu_2]}{}^{\sigma_1\sigma_2\sigma_3\sigma_4}F_{\sigma_1\sigma_2\sigma_3\sigma_4} + \frac{1}{3\cdot 3!}\Sigma^{ij}{}_{\nu_1\nu_2}{}^{\sigma_1\sigma_2\sigma_3}F_{\mu\sigma_1\sigma_2\sigma_3} \\
&- \frac{1}{3\cdot 3!}\Sigma^{ij}{}_{\mu[\nu_1}{}^{\sigma_1\sigma_2\sigma_3}F_{\nu_2]\sigma_1\sigma_2\sigma_3} + \frac{1}{3}K^{ij\ \sigma}F_{\sigma\mu\nu_1\nu_2} \\
\nabla_\mu \Sigma^{ij}_{\nu_1\nu_2\nu_3\nu_4\nu_5} &= \frac{1}{6}K^{ij\ \sigma} * F_{\sigma\mu\nu_1\nu_2\nu_3\nu_4\nu_5} - \frac{10}{3}F_{\mu[\nu_1\nu_2\nu_3}\Omega^{ij}_{\nu_4\nu_5]} - \frac{5}{6}F_{[\nu_1\nu_2\nu_3\nu_4}\Omega^{ij}_{\nu_5]\mu} \\
&- \frac{10}{3}g_{\mu[\nu_1}\Omega^{ij}{}_{\nu_2}{}^{\sigma}F_{|\sigma|\nu_3\nu_4\nu_5]} + \frac{5}{6}F_{\mu[\nu_1|\sigma_1\sigma_2|}(*\Sigma^{ij})^{\sigma_1\sigma_2}{}_{\nu_2\nu_3\nu_4\nu_5]} \\
&+ \frac{5}{6}F_{[\nu_1\nu_2|\sigma_1\sigma_2|}(*\Sigma^{ij})^{\sigma_1\sigma_2}{}_{\nu_3\nu_4\nu_5]\mu} - \frac{5}{9}g_{\mu[\nu_1}F_{\nu_2|\sigma_1\sigma_2\sigma_3|}(*\Sigma^{ij})^{\sigma_1\sigma_2\sigma_3}{}_{\nu_3\nu_4\nu_5]}
\end{aligned}
\tag{2.16}
$$

The exterior derivatives of the forms are thus given by

$$dK^{ij} = \frac{2}{3}i_{\Omega^{ij}}F + \frac{1}{3}i_{\Sigma^{ij}} * F \tag{2.17}$$

$$d\Omega^{ij} = i_{K^{ij}}F \tag{2.18}$$

$$d\Sigma^{ij} = i_{K^{ij}} * F - \Omega^{ij}\wedge F \tag{2.19}$$

where eg $(i_\Omega F)_{\mu\nu} = (1/2!)\Omega^{\rho_1\rho_2}F_{\rho_1\rho_2\mu\nu}$.

From the first equation in (2.16) we can immediately deduce the important result that each of the K^{ij} are Killing vectors. Moreover, using the Bianchi identity, it is simple to show that

$$\mathcal{L}_{K^{ij}}F = 0 \tag{2.20}$$

for any K^{ij}. Thus any geometry (g, F) admitting Killing spinors posesses symmetries generated by K^{ij}.

Notice, as somewhat of an aside, that using (2.17) we also have

$$\mathcal{L}_{K^{ij}} * F = i_{K^{ij}}(d * F + \frac{1}{2}F\wedge F) \tag{2.21}$$

Now the fact that K^{ij} is Killing and the condition (2.20) implies that both the left and right hand side must vanish separately. This means that the presence of a Killing spinor implies that some components of the equation of motion for the four-form are automatically satisfied[3]. Notice also that this calculation provides a check on the sign appearing in the Chern-Simons term in the D=11 supergravity Lagrangian, given the form of the Killing spinor equation and the conventions for the Clifford algebra.

[3]Note that we are ignoring the X_8 correction here. To fully consistently incorporate it one needs to also consider higher order correction terms to the Killing spinor equation. However, we note here that if $i_{K^{ij}}X_8 = 0$ the above equation is consistent.

We next note that (2.18) is strikingly similar to the notion of generalised calibration for static membranes introduced in [12] following [11]. Indeed consider the special case that $i = j$ and when $K = K^{ii}$ is a static Killing vector. Then taking into account (2.11) we see that (2.18) is exactly the same equation satisfied by a generalised calibration Ω for a membrane that was introduced in [12]. Recall that the significance of generalised calibrations is that they calibrate supersymmetric brane world-volumes in the presence of non-zero four-form flux. What we have shown here is that supersymmetric D=11 geometries automatically give rise to generalised calibrations Ω.

That one gets the same result, in this special case, either from D=11 supergravity or from the world-volume theory using kappa-symmetry as in [12], is perhaps not that surprising since it is well known that the kappa-symmetry of the super-membrane implies the equations of motion of D=11 supergravity [23, 24]. What is particularly inteteresting, though, is that the D=11 supergravity result indicates that the notion of generalised calibrations might be extended to more general settings than that studied in [12]. Firstly, since (2.18) is valid when K is not only static but more generally stationary, it suggests that the analysis of [12] can be straightforwardly extended to the stationary case, as assumed in that paper. Secondly, (2.18) is also valid when when K is null[4] and it should be very interesting to elucidate the physical interpretation of this from the world-volme point of view. Finally, when there is more than one Killing spinor, K^{ij} with $i \neq j$ can also be spacelike. This latter case is at least partially related to the issue discussed at the end of section II of [12] concerning the fact that static supersymmetric branes can have some flat directions.

The notion of generalised calibrations for fivebranes is more complicated due to the fact that the fivebrane world-volume has a self-dual three-form, which is responsible for the fact that membranes can end on fivebranes. An initial investigation was undertaken in [13] for the case of static configurations, where it was argued that the generalised calibration for the five-brane is a pair consisting of a spatial five-form and two-form. For static K^{ii} these correspond to the spatial part of Σ and Ω. The possibility of the five-form not being closed was considered in [13] and argued to be related to Wess-Zumino terms in the fivebrane worldvolume theory. That Ω might also not be closed was not considered in [13], but here we see that this is the general situation and it is not difficult to see that this is again related to Wess-Zumino terms in the fivebrane worldvolume theory. Moreover, our analysis reveals the correct differential expressions when K^{ii} is stationary and also when it is null, the latter case

[4]We will discuss later why spacelike K^{ii} are not possible.

again being particularly intriguing. As for membranes, K^{ij} with $i \neq j$ can also be spacelike when there is more than one Killing spinor.

It is also useful to note that we can also extract

$$
\begin{aligned}
(*d*\Omega^{ij})_{\nu} &= -\frac{1}{3\cdot 4!}(\Sigma^{ij})_{\nu}{}^{\sigma_1\sigma_2\sigma_3\sigma_4}F_{\sigma_1\sigma_2\sigma_3\sigma_4} \\
(*d*\Sigma^{ij})_{\nu_1\nu_2\nu_3\nu_4} &= \frac{8}{3}(\Omega^{ij})^{\sigma}{}_{[\nu_1}F_{\nu_2\nu_3\nu_4]\sigma} + \frac{2}{9}(*\Sigma^{ij})^{\sigma_1\sigma_2\sigma_3}{}_{[\nu_1\nu_2\nu_3}F_{\nu_4]\sigma_1\sigma_2\sigma_3} \qquad (2.22)
\end{aligned}
$$

Finally, using the algebraic results of the last subsection it is simple to conclude that the Lie-derivative of Ω and Σ with respect to K vanish:

$$
\begin{aligned}
\mathcal{L}_K\Omega &= 0 \\
\mathcal{L}_K\Sigma &= 0 \qquad (2.23)
\end{aligned}
$$

The corresponding equations for X, Y and Z are given by

$$
\begin{aligned}
\nabla_\mu X^{ij} &= -\frac{1}{3\cdot 3!}(Y^{ij})^{\rho_1\rho_2\rho_3}F_{\rho_1\rho_2\rho_3\mu_1} \\
\nabla_\mu Y^{ij}_{\nu_1\nu_2\nu_3} &= -\frac{1}{3}X^{ij}F_{\mu\nu_1\nu_2\nu_3} - \frac{1}{6\cdot 3!}Y^{ij\,\rho_1\rho_2\rho_3} * F_{\rho_1\rho_2\rho_3\mu\nu_1\nu_2\nu_3} \\
&- \frac{1}{4}Z^{ij\,\rho_1\rho_2}{}_{\mu[\nu_1}F_{\nu_2\nu_3]\rho_1\rho_2} - \frac{1}{2}Z^{ij\,\rho_1\rho_2}{}_{[\nu_1\nu_2}F_{\nu_3]\mu\rho_1\rho_2} - \frac{1}{6}g_{\mu[\nu_1}Z^{ij\,\rho_1\rho_2\rho_3}_{\nu_2}F_{\nu_3]\rho_1\rho_2\rho_3} \\
\nabla_\mu Z^{ij}_{\nu_1\nu_2\nu_3\nu_4} &= \frac{2}{3}Y^{ij}{}_{\mu[\nu_1}{}^{\rho}F_{\nu_2\nu_3\nu_4]\rho} - 2Y^{ij}{}_{[\nu_1\nu_2}{}^{\rho}F_{\nu_3\nu_4]\mu\rho} \\
&- g_{\mu[\nu_1}Y^{ij}{}_{\nu_2}{}^{\rho_1\rho_2}F_{\nu_3\nu_4]\rho_1\rho_2} - \frac{1}{9} * Z^{ij}{}_{\mu[\nu_1\nu_2\nu_3}{}^{\rho_1\rho_2\rho_3}F_{\nu_4]\rho_1\rho_2\rho_3} \\
&+ \frac{1}{18} * Z^{ij}{}_{\nu_1\nu_2\nu_3\nu_4}{}^{\rho_1\rho_2\rho_3}F_{\mu\rho_1\rho_2\rho_3} + \frac{1}{36}g_{\mu[\nu_1} * Z^{ij}{}_{\nu_2\nu_3\nu_4]}{}^{\rho_1\rho_2\rho_3\rho_4}F_{\rho_1\rho_2\rho_3\rho_4}
\end{aligned}
\qquad (2.24)
$$

and

$$
\begin{aligned}
(dX^{ij})_{\mu_1} &= -\frac{1}{3\cdot 3!}(Y^{ij})^{\rho_1\rho_2\rho_3}F_{\rho_1\rho_2\rho_3\mu_1} \\
(dY^{ij})_{\mu_1\mu_2\mu_3\mu_4} &= -\frac{1}{9}(Y^{ij})^{\rho_1\rho_2\rho_3}(*F)_{\rho_1\rho_2\rho_3\mu_1\mu_2\mu_3\mu_4} + (Z^{ij})_{[\mu_1\mu_2}{}^{\rho_1\rho_2}F_{\mu_3\mu_4]\rho_1\rho_2} \\
&- \frac{4}{3}X^{ij}F_{\mu_1\mu_2\mu_3\mu_4} \\
(dZ^{ij})_{\mu_1\mu_2\mu_3\mu_4\mu_5} &= -\frac{20}{3}(Y^{ij})_{[\mu_1\mu_2}{}^{\rho_1}F_{\mu_3\mu_4\mu_5]\rho_1} - \frac{5}{18}(*Z^{ij})_{[\mu_1\mu_2\mu_3\mu_4}{}^{\rho_1\rho_2\rho_3}F_{\mu_5]\rho_1\rho_2\rho_3}
\end{aligned}
\qquad (2.25)
$$

and also

$$
\begin{aligned}
(*d*Y^{ij})_{\nu_1\nu_2} &= 0 \\
(*d*Z^{ij})_{\nu_1\nu_2\nu_3} &= -\frac{1}{2}(Y^{ij})^{\sigma_1\sigma_2}{}_{[\nu_1}F_{\nu_2\nu_3]\sigma_1\sigma_2} - \frac{1}{36}(*Z^{ij})_{\nu_1\nu_2\nu_3}{}^{\sigma_1\sigma_2\sigma_3\sigma_4}F_{\sigma_1\sigma_2\sigma_3\sigma_4}
\end{aligned}
\qquad (2.26)
$$

2.3 Integrability

The integrability of the Killing spinor equation allows us to relate geometries admitting Killing spinors to those that in addition solve the equations of motion. As shown in the appendix, integrability of the Killing spinor equation implies that

$$\begin{aligned} 0 &= [R_{\mu\nu} - \frac{1}{12}(F_{\mu\sigma_1\sigma_2\sigma_3}F_{\nu}{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{12}g_{\mu\nu}F^2)]\Gamma^\nu\epsilon^i \\ &- \frac{1}{6\cdot 3!} * (d*F + \frac{1}{2}F\wedge F)_{\sigma_1\sigma_2\sigma_3}(\Gamma_\mu{}^{\sigma_1\sigma_2\sigma_3} - 6\delta_\mu^{\sigma_1}\Gamma^{\sigma_2\sigma_3})\epsilon^i \\ &- \frac{1}{6!}dF_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}(\Gamma_\mu{}^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} - 10\delta_\mu^{\sigma_1}\Gamma^{\sigma_2\sigma_3\sigma_4\sigma_5})\epsilon^i \end{aligned} \tag{2.27}$$

for each Killing spinor ϵ^i.

Assume that we have a geometry (g, F) that admits Killing spinors and that also solves the equations of motion and the Bianchi identity for F. We then deduce that

$$0 = E_{\mu\nu}\Gamma^\nu\epsilon^i = 0 \tag{2.28}$$

where $E_{\mu\nu} = 0$ is equivalent to the Einstein's equations. We now follow the analysis of [7]. Hitting this with $\bar{\epsilon}^i$ we conclude that

$$E_{\mu\nu}K^\nu = 0 \tag{2.29}$$

On the other hand if we hit it with $E_{\mu\sigma}\Gamma^\sigma$ we conclude that

$$E_{\mu\nu}E_\mu{}^\nu = 0 \qquad \text{no} \quad \text{sum} \quad \text{on} \quad \mu \tag{2.30}$$

As we shall discuss, $K \equiv K^{ii}$ is either a timelike or null Killing vector. We first assume that it is timelike. Introducing an orthonormal frame with $K = e^0$, we deduce from (2.29) that $E_{\mu 0} = 0$. The indices in (2.30) then run over spatial indices only and we conclude that $E_{\mu\nu} = 0$ since there are no non-trivial null vectors in a euclidean space. Alternatively if K is null we can set up a D=11 frame

$$ds^2 = 2e^+e^- + e^a e^a \tag{2.31}$$

for $a = 1, \ldots 9$, with $K = e^+$. Now (2.29) implies $E_{-\mu} = 0$ while (2.30) implies $E_{+a} = E_{ab} = 0$. Hence, one just needs to impose $E_{++} = 0$ to obtain a full supersymmetric solution.

These results have some obvious practical benefits in finding explicit solutions.

3 Classifying solutions using G-structures

In the last section we derived a number of necessary conditions, both algebraic and differential, for a geometry to posses Killing spinors. A useful organisational principle is that of a G-structure.

Let us begin by recalling the definition of G-structure of a n-dimensional manifold M. The frame bundle is a principal $Gl(n)$ bundle and a G-structure is simply a principle G-sub-bundle. Often the G-structure can be equivalently specified by the existence of no-where vanishing G-invariant tensors, and it is in this guise that G-structures often appear in the physics literature. For example, a metric of euclidean signature gives rise to an $O(n)$ structure and if supplemented with an orientation gives an $SO(n)$ structure. An almost complex structure J gives a $Gl(n/2, C)$ structure, and if supplemented with an hermitian metric gives a $U(n/2)$ structure, and so on. In D=11 supergravity the manifolds are equipped with a Lorentzian metric, and a spin structure, so the frame bundle can always be reduced to $Spin(1, 10)$ and hence there is always an $Spin(1, 10)$ structure.

Let us explain the main ideas in classifying G-structures using $G \subset Spin(1, 10)$ as an example (see e.g. [25, 26, 27] for further discussion). Consider a $G \subset Spin(1, 10)$ structure specified by G-invariant tensors and/or spinors, that we collectively define by η. The essential idea is simple: one takes the covariant derivative of η with respect to the Levi-Civita connection and then decomposes the result into irreducible G-modules. In more detail, one first uses the fact that there is no obstruction to finding a connection preserving the structure. If we choose one, ∇', then one notes that $\nabla\eta = (\nabla - \nabla')\eta$. Now $(\nabla - \nabla')$ is a tensor with values in $T^* \otimes spin(1, 10)$ but acting on the G-invariant η we see that the piece of $\nabla\eta$ that is independent of ∇' is given by an element of $T^* \otimes g^\perp$ where $g \oplus g^\perp = spin(1, 10)$. This element is known as the intrinsic torsion and can be decomposed into irreducible G-modules: $\nabla\eta \leftrightarrow T^* \otimes g^\perp = \oplus_i \mathcal{W}_i$. In one extreme, all of these modules $\mathcal{W}_i$ are present and one has the most general type of G-structure. In the other extreme, all of the modules vanish and the tensors are covariantly constant giving rise to manifolds with special holonomy G.

We can now use this language to interpret the algebraic and differential conditions that we obtained for Killing spinor bi-linears in the last section. In particular, it will provide us with a framework for classifying all supersymmetric solutions of D=11 supergravity.

Start with a D=11 geometry with a $Spin(1, 10)$ structure. The existence of a no-where vanishing Killing spinor then implies that the structure group can be further reduced to the isotropy group of the spinor. For a single spinor the isotropy group is known to be either $SU(5)$ or $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ [28]. These structures are in fact equivalently specified by the tensors K, Ω, Σ constructed from the bi-linears in the Killing spinors that we introduced in the last section. As shown in [28] the structure group is $SU(5)$ when the vector K is time-like and $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ when it is null. A spacelike K is not possible. The content of the algebraic conditions that we derived from Fierz identities in the last section is simply to ensure that K, Ω, Σ do indeed define the appropriate structure. In the next section we will analyse the $D = 11$ $SU(5)$ structure arising in the timelike case in more detail. A description of the more unusual $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure arising in the null case can be found in [28] (see also [29]).

So any geometry admitting a D=11 Killing spinor will either have an $SU(5)$ structure or a $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure. If the geometry admits further Killing spinors, there will be more than one of these structures or equivalently, the structure group can be reduced further. In each case the resulting structure will be given by the isotropy group of the spinors. So one part of classifying all geometries admitting Killing spinors is to classify all of the different isotropy groups of 1,...,32 spinors. In principle, one way of tackling this classification problem would be to derive algebraic conditions on the tensors $K^{ij}, \Omega^{ij}, \Sigma^{ij}$ and X^{ij}, Y^{ij}, Z^{ij} using various Fierz identities. However, the calculations we carried out for the tensors with $i = j$ in the last section were already very involved and this would be a very clumsy approach. It should be more efficient to generalise the work of [28].

A second aspect of classifying geometries admitting Killing spinors is to determine the types of G-structure that arise according to the classification of G-structures and also to see how much of the form of the geometry is specified by the structure. These can be determined by analysing the differential conditions imposed on the tensors $K^{ij}, \Omega^{ij}, \Sigma^{ij}$ and X^{ij}, Y^{ij}, Z^{ij} that we obtained from the Killing spinor equation. If one imposes the restriction that $F = 0$, one is then looking for Ricci flat manifolds with covariantly constant spinors. The classification is then a subset of that of special holonomy, which, as we noted are very special types of G-structure. A discussion of Lorentzian special holonomy can be found in [28, 29].

When $F \neq 0$ things are much more complicated. We will analyse this in detail in the next section for the case of a single timelike Killing spinor. We will see that the differential conditions restrict the type of $SU(5)$ structure. In addition we will be able to show that much, but not all, of the four-form field strength is in fact determined by the $SU(5)$ structure. We will also prove a converse result, i.e. that given such a $SU(5)$ structure, with the appropriately specified four-form, then the geometry does indeed admit at least one Killing spinor.

As noted in the last section, for the timelike case the integrability conditions for the Killing spinor imply that in order to have a supersymmetric solution to the equations of motion, one just needs to impose that the Bianchi identity and the equations of motion for the four-form are satisfied. These conditions impose further independent constraints.

A similar analysis for the null case, which we will leave for future work, would then provide a classification of the most general types of D=11 supergravity solutions. A finer classification using the G-structures that arise when there are more than one Killing spinor, would then complete the classification that we are advocating.

4 The stationary case and $SU(5)$ structure

In this section we will analyse solutions admitting at least one timelike Killing spinor. The spinor can be used to construct a one-form K, a two form Ω and a five-form Σ, which together specify an $SU(5)$ structure in D=11. An important restriction on this D=11 structure is that the dual time-like vector field to K is Killing. We can thus introduce a time coordinate along the orbits of the Killing vector, so that we have $K = -\Delta^2(dt + \omega)$, with Δ and ω independent of t. The metric then takes the form:

$$ds_{11}^2 = -\Delta^2(dt + \omega)^2 + \Delta^{-1} g_{mn} dx^m dx^n \tag{4.1}$$

and $K^2 = -\Delta^2$. The metric $\Delta^{-1} g_{mn}$ is a metric on the ten dimensional euclidean spatial base manifold, which we will denote by B, defined via the orthogonal projection of the eleven dimensional metric with respect to the Killing vector.

From (2.11) and (2.23) we immediately deduce that Ω is a two-form on the base manifold. If we raise an index using the metric g we obtain an almost complex structure on B. The metric g is then hermitian with respect to this almost complex structure and Ω is the Kähler form.

Using (2.12) it follows that the five-form Σ can be written as

$$\Sigma = \frac{1}{2}\Delta^{-1} e^0 \wedge \Omega \wedge \Omega + \Delta^{-3/2}\chi \tag{4.2}$$

where, again using (2.23), χ is a five-form on B and we have defined

$$e^0 = \Delta(dt + \omega) \tag{4.3}$$

which can be used to build an orthonormal frame in $D = 11$. If we now define

$$\theta = \chi - i * \chi \tag{4.4}$$

where, in this section, $*$ is the Hodge star with respect to the metric g, we conclude from (2.13) that θ is a $(5,0)$ form on B. This means that the ten-dimensional base manifold B admits an $SU(5)$ structure specified by Ω, θ, or equivalently by g, Ω, χ. In most of the subsequent analysis, the focus will be on the D=10 $SU(5)$ structure on B.

Note that the factors of Δ were inserted in the definition of χ in (4.2) to ensure that the $SU(5)$ structure satisfies the compatibility condition

$$\chi \wedge *\chi = -2^4 \frac{\Omega^5}{5!} \tag{4.5}$$

Note also that

$$\chi^2 = 16 \tag{4.6}$$

where indices here are contracted using the metric g, which can be deduced from, for example, (2.8).

The existence of an $SU(5)$ structure allows us to decompose the complexified space of forms on B into irreducible representations of $SU(5)$, and this will be very useful in the following. We first decompose the space of forms into (p,q)-forms. Pure forms of type $(p,0)$ form irreducible representations of $SU(5)$. For mixed forms we need to remove traces taken with Ω to form irreducible representations so these split further into:

$$\begin{aligned}
\Lambda^{(1,1)} &\cong \Lambda_0^{(1,1)} \oplus \mathbb{R} \\
\Lambda^{(2,1)} &\cong \Lambda_0^{(2,1)} \oplus \Lambda^{(1,0)} \\
\Lambda^{(2,2)} &\cong \Lambda_0^{(2,2)} \oplus \Lambda_0^{(1,1)} \oplus \mathbb{R} \\
\Lambda^{(3,1)} &\cong \Lambda_0^{(3,1)} \oplus \Lambda^{(2,0)} \\
\Lambda^{(3,2)} &\cong \Lambda_0^{(3,2)} \oplus \Lambda_0^{(2,1)} \oplus \Lambda^{(1,0)} \\
\Lambda^{(4,1)} &\cong \Lambda_0^{(4,1)} \oplus \Lambda^{(3,0)}
\end{aligned} \tag{4.7}$$

where the subscript 0 denotes a traceless form. The rest are determined by complex conjugation and by noting that $*$ maps a (p,q)-form to $(5-q, 5-p)$-form.

It will be helpful at this point to briefly review the classification of $SU(5)$ structures on ten-dimensional Riemannian manifolds (further comments are made in appendix C). We noted in the last section that G-structures are classified by the intrinsic torsion, which is an element of $T^* \otimes g^\perp$. Here $g^\perp$ is defined by $su(5) \oplus g^\perp \cong so(10)$. Noting that the adjoint of $so(10)$ decomposes under $su(5)$ via $\mathbf{45} \to \mathbf{1} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{24}$ and that $\mathbf{24}$ is the adjoint of $su(5)$, we conclude that the intrinsic torsion is given by the $SU(5)$ modules:

$$(\mathbf{5}+\bar{\mathbf{5}}) \times (\mathbf{1}+\mathbf{10}+\bar{\mathbf{10}}) \to (\mathbf{10}+\bar{\mathbf{10}}) + (\mathbf{40}+\bar{\mathbf{40}}) + (\mathbf{45}+\bar{\mathbf{45}}) + (\mathbf{5}+\bar{\mathbf{5}}) + (\mathbf{5}'+\bar{\mathbf{5}}') \quad (4.8)$$

In other words, the intrinsic torsion is given by five $SU(5)$ modules: $T^* \otimes g^\perp \simeq \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$, where conventionally[5] the $\mathcal{W}_i$ are given in the order noted in (4.8). The component of the intrinsic torsion in the module $\mathcal{W}_i$ will be denoted by W_i.

It will be very important in the following to use the fact that the W_i, and hence the intrinsic torsion, are determined by $d\Omega$ and $d\chi$. One sees that this is possible by consideration of the $su(5)$ irreps appearing in $d\Omega$ and $d\chi$. Consider first the three-form $d\Omega$ corresponding to the $\mathbf{120}$ of $SO(10)$. Since Ω is a $(1,1)$ form, $d\Omega$ will have a $(3,0)+(0,3)$ piece and also a $(2,1)+(1,2)$ piece. Removing the trace from the latter pieces, one obtains the decomposition $\mathbf{120} \to \mathbf{45} + \bar{\mathbf{45}} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{5} + \bar{\mathbf{5}}$ under $SU(5) \subset SO(10)$. Similarly, since χ is the real part of a $(5,0)$ form, the six-form $d\chi$ will have a $(5,1)+(1,5)$ and a $(4,2)+(2,4)$ part. These give rise to the representations $\mathbf{5} + \bar{\mathbf{5}} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{40} + \bar{\mathbf{40}}$. We thus see that $d\Omega$ and $d\chi$ contain all the irreps appearing in the W_i. In more detail we can define the following irreducible components of $d\Omega$ and $d\chi$:

$$\begin{aligned}
\chi \wedge d\Omega = \Omega \wedge d\chi &= W_1 \wedge \tfrac{\Omega^3}{3!} \\
(d\chi)^{4,2} + (d\chi)^{2,4} &= W_2 \wedge \Omega + \tfrac{1}{3} W_1 \wedge \tfrac{\Omega^2}{2!} \\
(d\Omega)^{2,1} + (d\Omega)^{1,2} &= W_3 + \tfrac{1}{4} W_4 \wedge \Omega \\
W_4 &= \Omega \lrcorner d\Omega \\
W_5 &= \chi \lrcorner d\chi
\end{aligned} \quad (4.9)$$

with $W_1 = *(\Omega \wedge d\chi)$. Here we have introduced the notation $\omega \lrcorner \nu$ which contracts a p-form ωinto a $n+p$-form ν via:

$$(\omega \lrcorner \nu)_{i_1 \ldots i_n} = \frac{1}{p!} \omega^{j_1 \ldots j_p} \nu_{j_1 \ldots j_p i_1 \ldots i_n} \quad (4.10)$$

A more precise connection between the intrinsic torsion and the W_i is presented in appendix C. Note that the $\mathbf{10} + \bar{\mathbf{10}}$ part of $d\chi$ is related to the $\mathbf{10} + \bar{\mathbf{10}}$ of $d\Omega$ through

[5] The modules $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ are those arising in the classification of $U(5)$ structures [30].

the condition $\Omega \wedge \chi = 0$. For orientation, note that the almost complex structure is integrable iff $W_1 = W_2 = 0$ so that manifolds with an $SU(5)$-structure of type $\mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$ are hermitian manifolds. Also if all W_i vanish so that the intrinsic torsion vanishes then the manifold is Ricci flat and has holonomy $G \subseteq SU(5)$. We will see that the $SU(5)$ structure arising on the base manifold B is only weakly restricted in general.

We are now ready to relate the components of F to the $SU(5)$-structure (g, Ω, χ). We will see that almost all of F is determined by the structure. First we write

$$F = \Delta^{-1} e^0 \wedge G + H \tag{4.11}$$

where G is a three-form and H is a four-form defined on B. The eleven dimensional Hodge dual of F is thus given by:

$$*_{11} F = \Delta^{-3} * G + \Delta^{-1} e^0 \wedge *H$$

We find from (2.18) that

$$G = d\Omega \tag{4.12}$$

and from (2.19) that

$$d(\Delta^{-3/2}\chi) + \frac{1}{2} d\omega \wedge \Omega \wedge \Omega = *H - \Omega \wedge H \tag{4.13}$$

So G is clearly determined by the structure. We now attempt to solve (4.13) for H. Introducing the map

$$\begin{aligned} \Theta : \Lambda^4(B) &\to \Lambda^4(B) \\ \alpha &\mapsto \alpha - *(\Omega \wedge \alpha) \end{aligned} \tag{4.14}$$

we can rewrite (4.13) as:

$$*\Theta(H) = d(\Delta^{-3/2}\chi) + \frac{1}{2} d\omega \wedge \Omega \wedge \Omega \tag{4.15}$$

Now H is a four-form and can be split into irreducible $SU(5)$ representations. The **210** of $SO(10)$ decomposes under $SU(5)$ as,

$$\mathbf{210} \to \mathbf{1} + \mathbf{5} + \bar{\mathbf{5}} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{24} + \mathbf{40} + \bar{\mathbf{40}} + \mathbf{75}$$

We can thus write

$$H = H_1 + H_{5+\bar{5}} + H_{10+\bar{10}} + H_{24} + H_{40+\bar{40}} + H_{75} \tag{4.16}$$

Irreps	E-Values
$\mathbf{1}$	-2
$\mathbf{5}, \bar{\mathbf{5}}$	0
$\mathbf{10}, \bar{\mathbf{10}}$	-1
$\mathbf{24}$	3
$\mathbf{40}, \bar{\mathbf{40}}$	2
$\mathbf{75}$	0

Table 1: Eigenvalues of the irreducible representations of Λ^6 under the map Θ. Since the map is real the conjugates of the complex representations have the same eigenvalues.

Each four-form H_i can then be written in terms of certain (p,q) forms defining $SU(5)$ irreps. Using the identities listed in (D.1) a calculation shows that each irreducible representation is an eigenvector of Θ with eigenvalues given in table 1. Note that three of the representations have zero eigenvalue. As we will see this will have important consequences. We next split the right hand side of (4.15) into irreducible $SU(5)$ components. For $d\chi$ this was noted above. For the second term we write $d\omega = d\omega^{(0)}\Omega + d\omega_0^{(1,1)} + d\omega^{(2,0)} + d\omega^{(0,2)}$, corresponding to the decomposition $\mathbf{45} \to \mathbf{1} + \mathbf{24} + \mathbf{10} + \bar{\mathbf{10}}$.

So let us what we can conclude from the above. First consider the $\mathbf{75}$ part. This is projected out on the left hand side of (4.15) but is also not present on the right hand side. So we have no contradiction here. Next consider the $\mathbf{5} + \bar{\mathbf{5}}$ part. Again this is projected out by Θ but is generically present on the left hand side. So we conclude that the $(5,1)$ piece of $d(\Delta^{-3/2}\chi)$ vanishes. Equivalently, we conclude that the $(5,1)$ piece of $d\chi$ corresponding to W_5 is exact:

$$W_5 = -12\ d\log\Delta \tag{4.17}$$

For the remaining representations the eigenvalues are non-zero and (4.15) allows us to determine the corresponding H_i in terms of the structure. We find,

$$\begin{aligned}
H_1 &= -\frac{3}{4}(d\omega)^{(0)}\Omega^2 \\
H_{10+\bar{10}} &= -[\frac{1}{3} * (\Omega \wedge d(\Delta^{-3/2}\chi)) + (d\omega)^{(2,0)} + (d\omega)^{(0,2)}] \wedge \Omega \\
H_{24} &= -\frac{1}{3}(d\omega)_0^{(1,1)} \wedge \Omega \\
H_{40+\bar{40}} &= \frac{1}{2} * d(\Delta^{-3/2}\chi) - \frac{1}{6} * (\Omega \wedge d(\Delta^{-3/2}\chi)) \wedge \Omega
\end{aligned} \tag{4.18}$$

where Ω^n denotes the wedge product of n factors of Ω.

At this point both H_{75} and $H_{5+\bar{5}}$ are undetermined. However, we can now use the equation for dK in (2.17) to relate the $\mathbf{5}+\bar{\mathbf{5}}$ part of $d\Omega$ and H to Δ giving:

$$\frac{\Delta^{3/2}}{6\ 5!} * H_{mn_1\ldots n_5}\chi^{n_1\ldots n_5} = \partial_m log\Delta - \frac{1}{6}\Omega^{r_1r_2}(d\Omega)_{mr_1r_2}+ \tag{4.19}$$

Looking at equations (2.22) we find no further constraints.

So let us summarize what we have learned. Any geometry admitting a timelike Killing spinor can be written in the form

$$ds^2_{11} = -\Delta^2(dt+\omega)^2 + \Delta^{-1}g_{mn}dx^m dx^n \tag{4.20}$$

where the base space with metric g admits an $SU(5)$ structure (g,Ω,χ) whose only restriction is that W_5 is exact and related to the warp factor Δ via

$$W_5 = -12 d\log\Delta \tag{4.21}$$

The four-form field strength can be written as:

$$\begin{aligned} F &= (dt+\omega)\wedge d\Omega - [\frac{3}{4}(d\omega)^{(0)}\Omega + (d\omega)^{(2,0)} + (d\omega)^{(0,2)} + \frac{1}{3}(d\omega)^{(1,1)}_0]\wedge\Omega \\ &+ \frac{1}{2} * d(\Delta^{-3/2}\chi) - \frac{1}{2} * [\Omega\wedge d(\Delta^{-3/2}\chi)]\wedge\Omega \\ &- \frac{1}{16} * ([W_5 + 4W_4]\wedge\Delta^{-3/2}\chi) + F_{75} \end{aligned} \tag{4.22}$$

where F_{75} $(= H_{75})$ is an arbitrary 4-form on B in the $\mathbf{75}$ of $SU(5)$ (i.e. $F_{75}\in\Lambda^{(2,2)}_0$), W_4 and W_5 are defined in (4.9) and $d\omega = d\omega^{(0)}\Omega + d\omega^{(1,1)}_0 + d\omega^{(2,0)} + d\omega^{(0,2)}$.

We started this section with the $SU(5)$ structure in D=11 specified by K,Ω,Σ. However, our derivation of (4.20), (4.21), (4.22) mostly involved the $SU(5)$ structure in D=10 which is a component of the D=11 $SU(5)$ structure. The reason for this is that the D=11 structure is constrained by the fact that K is Killing, $\mathcal{L}_K\Omega = \mathcal{L}_K\Sigma = 0$ and we worked with the obvious adapted co-ordinates. Now $d\omega$ is an arbitrary closed[6] two-form on B as far as the D=10 SU(5) structure is concerned. On the other hand, one can show that dK specifies a part of the intrinsic torsion of the D=11 structure, and since $dK = 2d(log\Delta)\wedge K - \Delta^2 d\omega$ we conclude that $d\omega$ is in fact determined by the $D=11$ structure. This is in contrast to F_{75} which is determined by neither the D=10 nor the D=11 structure. See appendix E for further comments about the type of $SU(5)$ structure in D=11.

We have thus derived necessary conditions on the form of the most general geometry admitting a timelike Killing spinor. The form includes a completely undetermined

[6]Since ω need only be locally defined.

component of the field strength and one might wonder if further conditions might be imposed by considering the covariant derivatives of Ω and Σ and not just the exterior derivatives. To see that they are not we will now prove a converse result: that the above form of the metric and four-form field strength does indeed admit a Killing spinor. In particular F_{75} drops out of the Clifford connection appearing in the Killing spinor equation. It is for this reason that this component of the field-strength is not determined by the Killing spinor equation alone.

To see this we first note that the geometry should preserve a single Killing spinor ϵ giving rise to the $SU(5)$ structure in $D = 11$. Such a spinor can be specified by demanding that it be left inert by a number of projection operators. First introduce the obvious orthonormal frame

$$\begin{aligned} e^0 &= \Delta(dt + \omega) \\ e^i &= \Delta^{-1/2}\bar{e}^i \end{aligned} \tag{4.23}$$

where $\bar{e}^i$ is an orthonormal frame for the base manifold B. The D=11 gamma matrices give rise to D=10 gamma-matrices Γ^i with $\Gamma^0 = \Gamma_{1\ldots10}$ proportional to the chirality operator. It is convenient to introduce the chiral complex spinor

$$\eta = \frac{(1 + i\Gamma_0)}{\sqrt{2}}\epsilon \tag{4.24}$$

in terms of which

$$\begin{aligned} \Omega_{ij} &= -i\eta^\dagger\Gamma_{ij}\eta \\ \theta_{i_1\ldots i_5} &= i\eta^T\Gamma_{i_1\ldots i_5}\eta \end{aligned} \tag{4.25}$$

Both the real an imaginary parts of η give equivalent $SU(5)$ structures, but only the real part will be a Killing spinor, as we shall see. Now the complex structure on the base manifold is not integrable in general, and hence we cannot always introduce complex co-ordinates. Nevertheless we can consistently introduce holomorphic and anti-holomorphic tangent space indices which simplifies the calculation. In terms of these we conclude that η satisfies the projections

$$\Gamma^a\eta = 0 \tag{4.26}$$

where $a = 1\ldots5$ is a holomorphic index and also

$$\Gamma_{a_1\ldots a_5}\eta = -i\theta_{a_1\ldots a_5}\eta^* \tag{4.27}$$

We now consider the Killing spinor equation acting on $\epsilon = (\eta+\eta^*)/\sqrt{2}$. Plugging in the expression for the four-form and dealing with each $SU(5)$ irrep separately, we find after a lengthy computation and using (D.2), that

$$\begin{aligned}[\nabla_m &+ \frac{1}{160}(\Omega W_5 + 5\Omega W_4)_m \Omega_{k_1k_2}\Gamma^{k_1k_2} - \frac{1}{16}(W_4)_k \Gamma_m{}^k \\ &+ \frac{1}{8}\Omega_m{}^r (W_3)_{rk_1k_2}\Gamma^{k_1k_2} - \frac{1}{394}\chi_{mk_1k_2}{}^{n_1n_2}(W_1)_{n_1n_2}\Gamma^{k_1k_2} \\ &+ \frac{1}{192}\Omega_m{}^r (W_2)_{r\ell_1\ell_2\ell_3}\chi^{\ell_1\ell_2\ell_3}{}_{k_1k_2}\Gamma^{k_1k_2}](\eta_0+\eta_0^*) = 0 \end{aligned} \tag{4.28}$$

where we have rescaled the spinor $\eta \equiv \Delta^{1/2}\eta_0$ and used the notation $\Omega V_m \equiv \Omega_m{}^r V_r$. Now both η_0 and η_0^* are solutions to this equation since the connection is simply the sum of the Levi-Civita connection on B with the intrinsic contorsion of the $SU(5)$ structure, as we show in appendix C. However, one should not conclude that there are two Killing spinors: the point is that (4.28) only arises when the Killing spinor equation is acting on the sum $(\eta+\eta^*)/\sqrt{2}$ and not on η, η^* separately. Thus we conclude that the geometry in general preserves one Killing spinor $\Delta^{1/2}(\eta_0+\eta_0^*)$ corresponding to just 1/32 supersymmetry.

It is interesting to note that while the covariant derivative appearing in the original Killing spinor equation of D=11 supergravity (2.3) takes values in the Clifford algebra, the covariant derivative appearing in (4.28) takes values in the spin sub-algebra. In other words we have shown that the four-form field strength is necessarily constrained in such a way that it transforms the Clifford connection into a spin connection when acting on the preserved supersymmetries.

We have now shown that the form (4.20), (4.21), (4.22) is both necessary and sufficient for a geometry to admit a single time-like Killing spinor. However not all such spacetimes are solutions of eleven dimensional supergravity. To obtain solutions of the theory one just has to impose the gauge equations of motion and the Bianchi identity for F since the Einstein equations will then be automatically be satisfied as we showed in the last section. The Bianchi identity for F can be written

$$d\omega \wedge d\Omega + dH = 0 \tag{4.29}$$

while the equation of motion for the four-form gives rise to two equations

$$\begin{aligned} d(\Delta^{-3} * d\Omega) + d\omega \wedge *H + \frac{1}{2}H\wedge H &= (2\pi)^4 X_8 \\ d\Omega \wedge H - d * H &= 0 \end{aligned} \tag{4.30}$$

Here we have added in the correction term to the field equation and have assumed that $i_K X_8 = 0$. Note that the third equation is actually implied by the first (see the discussion following (2.21)). To check this in detail one can take the exterior derivative of (4.15) to find

$$d(*\Theta(H)) - d\omega \wedge d\Omega \wedge \Omega = 0 \tag{4.31}$$

and then substitute (4.29). Note that one can further substitute the expression for H given by (4.18) and (4.19) into (4.29), (4.30) but as the result is not too illuminating we shall not present it here. It is worth emphasising that the component of the four-form not determined by the Killing spinor equation, F_{75}, is constrained and related to the $SU(5)$ structure by the Bianchi identity and the equations of motion

At this stage we can present some vanishing theorems when the ten-dimensional base manifold B is compact. Consider first the case when $H = 0$ and hence $F = (dt + \omega) \wedge d\Omega$. We then have

$$\int_B \Delta^{-3} d\Omega \wedge * d\Omega = - \int_B \Omega \wedge d(*\Delta^{-3} d\Omega) = 0 \tag{4.32}$$

where we have integrated by parts and then used the equation of motion (4.30) (ignoring the X_8 correction). Since the left hand side of the equation is positive semi-definite we conclude that $d\Omega = 0$ which in turn implies that the four-form $F = 0$.

Let us now obtain another result for non-vanishing H. First observe that using (4.15) and (4.18) we can conclude

$$d(\Delta^{-3/2}\chi) = *\Theta(H) - \frac{1}{2} d\omega \wedge \Omega^2 = *\Theta(H') \tag{4.33}$$

where H' is defined to be the pieces of H that are independent of $d\omega$. Next consider

$$\int_B *\Theta(H') \wedge \Theta(H') = - \int_B \Delta^{-3/2} \chi \wedge d\Theta(H') \tag{4.34}$$

where we have integrated by parts. We next note that

$$d\Theta(H') = d(-H'_{10+\bar{10}} + 2H_{40+\bar{40}}) \tag{4.35}$$

On the other hand we know from the Bianchi identity (4.29) that

$$d\omega \wedge d\Omega + dH = 0 \tag{4.36}$$

If we now restrict to $d\omega = d\omega^{(2,0)} + d\omega^{(0,2)}$ then this equation becomes

$$d(H_{5+\bar{5}} - H'_{10+\bar{10}} + H_{40+\bar{40}} + H_{75}) = 0 \tag{4.37}$$

Comparing with (4.35) we further restrict $H_{5+\bar{5}} = H_{75} = 0$.

We now obtain our result. If $d\omega = d\omega^{(2,0)} + d\omega^{(0,2)}$, $H_{5+\bar{5}} = H_{75} = 0$ and $H_{40+\bar{40}} = 0$ then (4.37), (4.35) and (4.33) implies that $H_{10+\bar{10}} = 0$ also and hence H=0. Similarly if $d\omega = d\omega^{(2,0)} + d\omega^{(0,2)}$, $H_{5+\bar{5}} = H_{75} = 0$ and $H_{10+\bar{10}} = 0$ then $H_{40+\bar{40}} = 0$ also and hence H=0. In both cases the previous result assuming $H = 0$ and arbitrary $d\omega$ then implies that $F = 0$.

5 Examples of solutions with $SU(5)$ structures

In order to gain some further insight into the formalism, we will now display $SU(5)$ structures for some known solutions. As a bonus, in carrying out this exercise we will be able to spot some new solutions.

5.1 M5 branes

Let us first look at the simple $M5$-brane solution. The metric and field strength can be written as:

$$
\begin{aligned}
ds^2_{11} &= H^{-1/3}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2) + H^{2/3} dy^i dy^i \\
*_{11}F &= -dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dH^{-1}
\end{aligned}
\tag{5.1}
$$

with $i = 1, \ldots, 5$ and $H = H(y)$ a harmonic function. This solution is well known to preserve 1/2 of the supersymmetry: the Killing spinors satisfy the single projection $\Gamma_{012345}\epsilon = \epsilon$. There are certainly timelike spinors which satisfy this projection and so we should be able to display a $SU(5)$ structure for it.

Comparing with (4.20) we identify $\Delta = H^{-1/6}$ and the base space metric is then given by,

$$
ds^2 = H^{-1/2} dx^i dx^i + H^{1/2} dy^i dy^i \tag{5.2}
$$

Define the complex $(1, 0)$ frame,

$$
\Theta^i = H^{-1/4} dx^i + i H^{1/4} dy^i \tag{5.3}
$$

The corresponding $SU(5)$ structure is given by:

$$
\begin{aligned}
\Omega &= \frac{i}{2} \Theta^i \wedge \bar{\Theta}^i \\
\chi &= Re(\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^4 \wedge \Theta^5)
\end{aligned}
\tag{5.4}
$$

In terms of the coordinate basis these are given by

$$
\begin{aligned}
\Omega &= dx^i \wedge dy^i \\
\chi &= H^{-5/4} dx^{12345} - \frac{H^{-1/4}}{3!2!} \varepsilon_{i_1 \ldots i_5} dx^{i_1 i_2 i_3} \wedge dy^{i_4 i_5} + \frac{H^{3/4}}{4!} \varepsilon_{i_1 \ldots i_5} dx^{i_1} \wedge dy^{i_2 \ldots i_5}
\end{aligned}
\tag{5.5}
$$

where $\varepsilon_{i_1 \dots i_5}$ is just the $d = 5$ permutation symbol.

To see that this $SU(5)$ structure is indeed related to a Killing spinor we first note, after a small calculation, that $d(\Delta^{-3/2}\chi)$ has no $(5,1)+(1,5)$ pieces and hence (4.21) is satisfied. This is the only restriction required on the structure, but we note that here we also have $d\Omega = 0$. We next need to show that the four-form can be recovered from (4.22). Interestingly, to achieve this it is necessary to include a non-vanishing F_{75}. Specifically we set

$$F_{75} = \frac{1}{16 \cdot 2!2!} \partial^i log H \varepsilon_{ij_1 \dots j_4} \Theta^{j_1} \wedge \Theta^{j_2} \wedge \bar{\Theta}^{j_3} \wedge \bar{\Theta}^{j_4} \tag{5.6}$$

and then (4.22) agrees with the expression in (5.1).

Since the fivebrane solution preserves 16 Killing spinors the solution has more than one $SU(5)$ structure. Note also that some of the Killing spinors can be null so that the solution also belongs to the null class. It would be interesting to display the $SU(5)$ structure for the solution corresponding to a fivebrane wrapping a SLAG five-cycle [31], as this solution preserves just 1/32 supersymmetry .

5.2 Flat and resolved $M2$ branes

Let us now recover some known solutions involving membranes. We take the ten-dimensional base space B to be of the form:

$$ds^2 = \Delta^3(dx_1^2 + dx_2^2) + ds^2(M_8) \tag{5.7}$$

where $ds^2(M_8)$ is a Ricci flat metric with holonomy contained in $SU(4)$. One can then define the following $SU(5)$ structure

$$\begin{aligned} \Omega &= \Delta^3 dx^1 \wedge dx^2 + \omega_{(8)} \\ \chi &= \Delta^{3/2}(dx^1 \wedge \hat{\chi}_1 + dx^2 \hat{\chi}_2) \end{aligned} \tag{5.8}$$

where we have introduced the Kähler form $\omega_{(8)}$ and holomorphic $(4,0)$ form $\hat{\theta} = \hat{\chi}_1 - i\hat{\chi}_2$ of M_8. The base space has in fact an $SU(4) \subset SU(5)$ structure. Note that the normalizations of Ω and χ are not arbitrary but are chosen to ensure that Ω is a Kähler form for the base space and that they satisfy the compatibility condition (4.5). Also we could have chosen an arbitrary function f^2 in the metric instead of Δ^3 but demanding that (4.21) is satisfied implies that $f^2 = \Delta^3$.

For simplicity we will assume that Δ does not depend on (x^1, x^2). This implies that the $\mathbf{5} + \bar{\mathbf{5}}$ piece of the spatial part of the four-form field strength vanishes. Let us first restrict to static solutions and set the rotation parameter to zero. The expression for the four-form field strength (4.22) becomes

$$F = dt \wedge dx^1 \wedge dx^2 \wedge d\Delta^3 + F_{75} \tag{5.9}$$

where F_{75} is any four form on the base space in the **75** of $SU(5)$. Imposing the Bianchi identity and gauge equations of motion and using $*F_{75} = F_{75} \wedge \Omega$ one finds:

$$\begin{aligned} dF_{75} &= 0 \\ d *_8 d\Delta^{-3} - \frac{1}{2} F_{75} \wedge F_{75} &= (2\pi)^4 X_8 \end{aligned} \tag{5.10}$$

where $*_8$ is the Hodge star with respect to the 8 dimensional metric.

In the simple case that $F_{75} = 0$ and flat transverse space, we recover the well known 1/2 supersymmetric $M2$ brane solution

$$\begin{aligned} ds^2 &= H^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3} ds^2(\mathbb{E}^8) \\ F &= dt \wedge dx^1 \wedge dx^2 \wedge d(H^{-1}) \end{aligned} \tag{5.11}$$

where $H \equiv \Delta^{-3}$ is harmonic.

Another possibility is to take F_{75} to be a four form on M_8. Under $SU(4) \subset SU(5)$ we have the following decomposition: $\mathbf{75} \to \mathbf{15} + \mathbf{20} + \bar{\mathbf{20}} + \mathbf{20}'$. The first three representations occur when one of the indices of F_{75} is in the (x^1, x^2) directions so for F_{75} to be a four form on M_8 it must belong to the $\mathbf{20}'$ of $SU(4)$ i.e. it must be a self-dual $(2, 2)$ form. Since it must be closed it follows that $F_{75} = L_{(2,2)}$ with $L_{(2,2)}$ a harmonic self dual four form. This modifies the equation for H and we get

$$\Box H = -\frac{1}{2} |L_{(2,2)}|^2 - (2\pi)^4 X_8 \tag{5.12}$$

Thus we recover the resolved 1/8 supersymmetric $M2$ brane solutions of [18, 19, 20, 16, 17]. As for the fivebrane solution, these solutions also belong to the null class.

A simple rotating generalisation of these solutions is to choose $d\omega$ to be the sum of a $(2, 0) + (0, 2)$ form. Specifically, given a closed two-form $\alpha \in \Lambda^{2,0}(M_8)$ we set $d\omega = \alpha + \bar{\alpha}$ and get the supersymmetric solution

$$\begin{aligned} ds^2 &= H^{-2/3}[-(dt + \omega)^2 + dx_1^2 + dx_2^2] + H^{1/3} ds^2(M_8) \\ F &= (dt + \omega) \wedge dx^1 \wedge dx^2 \wedge d(H^{-1}) + F_{75} \end{aligned} \tag{5.13}$$

with (5.10) unchanged.

5.3 Rotating Calabi-Yau and the Gödel solution

Recently, an interesting generalisation of the Gödel solution was found in five-dimensions [7]. Uplifted to D=11 it was shown to preserve 5/8 of the supersymmetry. We now show that this fits into a broader class of new solutions.

We look for rotating solutions with no warp factor for which the base space M_{10} is a complex manifold with holonomy $G \subseteq SU(5)$. Then, the only non-zero components of the field strength can arise from the rotation and from F_{75}. Similarly to the last sub-section, we set $d\omega = \alpha + \bar{\alpha}$ with $\alpha \in \Lambda^{2,0}(M_{10})$. We then find the supersymmetric solution

$$\begin{aligned} ds^2_{11} &= -(dt+\omega)^2 + ds^2(M_{10}) \\ F &= -d\omega \wedge \Omega + F_{75} \end{aligned} \tag{5.14}$$

provided that $dF_{75} = 0$ and $F_{75} \wedge F_{75} = -2(2\pi)^4 X_8$.

As a particular example of this class of solutions we take the base space to be flat $\mathbb{E}^{10}$ and set $F_{75} = 0$. Introduce complex coordinates z^a for $\mathbb{E}^{10}$ and the canonical $SU(5)$ structure,

$$\begin{aligned} \Omega &= \frac{i}{2} dz^a \wedge d\bar{z}^a \\ \chi &= Re(dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5) \end{aligned} \tag{5.15}$$

One can then choose $\alpha = dz^1 \wedge dz^2$ and this gives the Gödel solution of [7] which preserves 5/8 supersymmetry. It would be interesting to see if any of the more general solutions with $\alpha \in \Lambda^{2,0}(\mathbb{E}^8)$ also preserve exotic fractions of supersymmetry.

6 Conclusion

We have shown that the most general supersymmetric configurations of D=11 supergravity, preserving at least one Killing spinor, have either an $SU(5)$ or an $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, depending on whether the vector constructed from a bi-linear of the Killing spinor is time-like or null, respectively. For the time-like case, we carried out a detailed analysis using the Killing spinor equation: we found that the $SU(5)$ structure in D=11 is restricted by the fact that the time-like vector is always Killing and that the $SU(5)$ structure of the D=10 base space orthogonal to the orbits of the Killing vector is only weakly constrained. We deduced the general form of the geometry admitting Killing spinors and showed that most of its form is determined by the D=11 $SU(5)$ structure. In particular there was a component of the four-form which dropped out of the Killing spinor equation and hence is undetermined. We also analysed what extra constraints are imposed in order to ensure that the geometries preserving Killing spinors also solve the equations of motion. These constraints relate the component of the four-form undetermined by the Killing spinor equation to the $SU(5)$ structure. To complete the classification of the most general supersymmetric solutions we need to carry out a similar analysis for the null case.

We have also proposed a finer classification for configurations that preserve more than one supersymmetry. Such configurations will have various numbers of different $SU(5)$ and/or $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures, or equivalently, a G-structure with $G \subset SU(5)$ or $G \subset (Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$. The first step then, is to classify these G-structures which are defined to be the isotropy groups of the Killing spinors. They can also be specified by algebraic conditions on the tensors that can be constructed from bi-linears in the spinors. The second step, in the classification is to then use the Killing spinor equation to place constraints on the G-structure, as well as to solve for various parts of the metric and four-form field strength in terms of the structure. It would be quite an achievement to carry out this programme in full.

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A Conventions

We use the signature $(-,+,...,+)$. D=11 co-ordinate indices will be denoted $\mu, \nu, \ldots$ while tangent space indices will be denoted by $\alpha, \beta, \ldots$.

The D=11 spinors we will use are Majorana. The gamma matrices satisfy

$$\{\Gamma_\alpha, \Gamma_\beta\} = 2\eta_{\alpha\beta} \tag{A.1}$$

and can be taken to be real in the Majorana representation. They satisfy, in our conventions, $\Gamma_{012345678910} = 1$ and as a consequence of this the following duality relation holds:

$$\Gamma_{\alpha_1\alpha_2\ldots\alpha_p} = (-1)^{\frac{(p+1)(p-2)}{2}} \frac{1}{(11-p)!} \varepsilon_{\alpha_1\alpha_2\ldots\alpha_p}{}^{\alpha_{p+1}\alpha_{p+2}\ldots\alpha_{11}} \Gamma_{\alpha_{p+1}\alpha_{p+2}\ldots\alpha_{11}} \tag{A.2}$$

where we have defined

$$\varepsilon_{012345678910} = +1 \tag{A.3}$$

It follows that $\Delta = \{1, \Gamma_{\alpha_1}, \Gamma_{\alpha_1\alpha_2}, \Gamma_{\alpha_1\alpha_2\alpha_3}, \Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}, \Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\}$ is a basis of the Clifford algebra $C\ell(10,1) \cong R(32)$, where $R(32)$ is the algebra of 32×32 matrices. We will use repeatedly the following formula for antisymmetrizing products of gamma matrices:

$$\Gamma_{\alpha_1\alpha_2\ldots\alpha_n}\Gamma^{\beta_1\beta_2\ldots\beta_m} = \sum_{k=0}^{k=min(n,m)} \frac{m!n!}{(m-k)!(n-k)!k!}\Gamma_{[\alpha_1\alpha_2\ldots\alpha_{n-k}}{}^{[\beta_{k+1}\ldots\beta_m}\delta^{\beta_1}_{\alpha_n}\delta^{\beta_2}_{\alpha_{n-1}}\ldots\delta^{\beta_k]}_{\alpha_{n-k+1}]} \tag{A.4}$$

For any $M, N \in R(32)$ we can perform a Fierz rearrangement using:

$$\begin{aligned} M_a{}^b N_c{}^c &= \frac{1}{32}\{(NM)_a{}^d\delta_c{}^b + (N\Gamma^{\alpha_1}M)_a{}^d(\Gamma_{\alpha_1})_c{}^b \\ &- \frac{1}{2!}(N\Gamma^{\alpha_1\alpha_2}M)_a{}^d(\Gamma_{\alpha_1\alpha_2})_c{}^b - \frac{1}{3!}(N\Gamma^{\alpha_1\alpha_2\alpha_3}M)_a{}^d(\Gamma_{\alpha_1\alpha_2\alpha_3})_c{}^b \\ &+ \frac{1}{4!}(N\Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4}M)_a{}^d(\Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4})_c{}^b + \frac{1}{5!}(N\Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}M)_a{}^d(\Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5})_c{}^b\} \end{aligned} \tag{A.5}$$

where $a, b, c, d = 1, \ldots, 32$.

Given a Majorana spinor ϵ its conjugate is given by $\bar{\epsilon} = \epsilon^T C$, where C is the charge conjugation matrix in D=11 and satisfies $C^T = -C$. In the Majorana representation we can choose $C = \Gamma_0$. An important property of gamma matrices in D=11 is that the matrix $C\Gamma_{\alpha_1\alpha_2\ldots\alpha_p}$ is symmetric for $p = 1, 2, 5$ and antisymmetric for $p = 0, 3, 4$ (the cases $p > 5$ are related by duality to the above). For an antisymmetrized product $\Gamma_{(n)}$ of n gamma matrices and any spinor ϵ we have :

$$\overline{\Gamma_{(n)}\epsilon} = (-1)^{\frac{n(n+1)}{2}}\bar{\epsilon}\Gamma_{(n)} \tag{A.6}$$

The Hodge star of a p-form ω is defined by

$$*\omega_{\mu_1\ldots\mu_{11-p}} = \frac{\sqrt{-g}}{p!}\epsilon_{\mu_1\ldots\mu_{11-p}}{}^{\nu_1\ldots\nu_p}\omega_{\nu_1\ldots\nu_p} \tag{A.7}$$

and the square of a p-form via

$$\omega^2 = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\omega^{\mu_1\ldots\mu_p} \tag{A.8}$$

B Integrability conditions from the Killing spinor equation

Taking the second covariant derivative of the Killing spinor equation (2.3) and antisymmetrising we obtain:

$$\nabla_{[\rho}\nabla_{\mu]}\epsilon = -\frac{1}{288}(\Gamma_{[\mu}{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta^{\nu_1}_{[\mu}\Gamma^{\nu_2\nu_3\nu_4})\nabla_{\rho]}F_{\nu_1\nu_2\nu_3\nu_4}\epsilon$$

$$+ \quad \frac{1}{288^2}(\Gamma_{[\mu}{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta^{\nu_1}_{[\mu}\Gamma^{\nu_2\nu_3\nu_4})(\Gamma_{\rho]}{}^{\sigma_1\sigma_2\sigma_3\sigma_4} - 8\delta^{\sigma_1}_{[\rho}\Gamma^{\sigma_2\sigma_3\sigma_4})F_{\nu_1\nu_2\nu_3\nu_4}F_{\sigma_1\sigma_2\sigma_3\sigma_4}\epsilon \tag{B.1}$$

The terms on the right hand side can be simplified using the identity:

$$\begin{gathered}(\Gamma_{[\mu}{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta^{\nu_1}_{[\mu}\Gamma^{\nu_2\nu_3\nu_4})(\Gamma_{\rho]}{}^{\sigma_1\sigma_2\sigma_3\sigma_4} - 8\delta^{\sigma_1}_{\rho]}\Gamma^{\sigma_2\sigma_3\sigma_4}) = \\ \Gamma_{\mu\rho}{}^{\nu_1\nu_2\nu_3\nu_4\sigma_1\sigma_2\sigma_3\sigma_4} + 16\delta^{\nu_1}_{[\mu}\Gamma_{\rho]}{}^{\nu_2\nu_3\nu_4\sigma_1\sigma_2\sigma_3\sigma_4} + 2\cdot 4!\delta^{\nu_1\sigma_1}_{\mu\rho}\Gamma^{\nu_2\nu_3\nu_4\sigma_2\sigma_3\sigma_4} \\ +4\cdot 4!\delta^{\nu_1}_{[\mu}g^{\nu_2\sigma_1}\Gamma_{\rho]}{}^{\nu_3\nu_4\sigma_2\sigma_3\sigma_4} - 3\cdot 4!g^{\nu_1\sigma_1}g^{\nu_2\sigma_2}\Gamma_{\mu\rho}{}^{\nu_3\nu_4\sigma_3\sigma_4} - 4!4!\delta^{\nu_1}_{[\mu}g^{\nu_2\sigma_1}g^{\nu_3\sigma_2}\Gamma_{\nu]}{}^{\nu_4\sigma_3\sigma_4} \\ +16\cdot 4!\delta^{\nu_1\nu_2}_{\mu\rho}g^{\nu_3\sigma_1}\Gamma^{\nu_4\sigma_2\sigma_3\sigma_4} + 4!g^{\nu_1\sigma_1}g^{\nu_2\sigma_2}g^{\nu_3\sigma_3}g^{\nu_4\sigma_4}\Gamma_{\mu\rho} - 8\cdot 4!\delta^{\nu_1}_{[\mu}g^{\nu_2\sigma_1}g^{\nu_3\sigma_2}g^{\nu_4\sigma_3}\Gamma_{\rho]}{}^{\sigma_4} \\ -36\cdot 4!\delta^{\nu_1\sigma_1}_{\mu\rho}g^{\nu_2\sigma_2}g^{\nu_3\sigma_3}\Gamma^{\nu_4\sigma_4} \qquad (AS)\end{gathered} \tag{B.2}$$

where "AS" refers to the fact that this equation is true when we anti-symmetrise over the indices $\sigma_1, \ldots, \sigma_4$ and $\nu_1, \ldots, \nu_4$. Also, the left hand side can be expressed in terms of the Riemann tensor via

$$\nabla_{[\rho}\nabla_{\mu]}\epsilon = \frac{1}{8}R_{\rho\mu\sigma_1\sigma_2}\Gamma^{\sigma_1\sigma_2}\epsilon \tag{B.3}$$

Now contracting both sides of this equation with Γ^μ and using the Bianchi identity $R_{\mu[\nu\rho\sigma]} = 0$ we find that:

$$\Gamma^\mu\nabla_{[\rho}\nabla_{\mu]}\epsilon = -\frac{1}{4}R_{\rho\mu}\Gamma^\mu \tag{B.4}$$

Evaluating the right hand side one finds the integrability condition:

$$\begin{aligned}0 \quad &= \quad (R_{\rho\mu} - \frac{1}{12}(F_{\rho\sigma_1\sigma_2\sigma_3}F_\mu{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{12}g_{\rho\mu}F^2))\Gamma^\mu\epsilon \\ &- \quad \frac{1}{6\cdot 3!} * (d*F + \frac{1}{2}F\wedge F)_{\sigma_1\sigma_2\sigma_3}(\Gamma_\rho{}^{\sigma_1\sigma_2\sigma_3} - 6\delta^{\sigma_1}_\rho\Gamma^{\sigma_2\sigma_3})\epsilon \\ &- \quad \frac{1}{6!}dF_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}(\Gamma_\rho{}^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} - 10\delta^{\sigma_1}_\rho\Gamma^{\sigma_2\sigma_3\sigma_4\sigma_5})\epsilon\end{aligned} \tag{B.5}$$

C $SU(5)$ structures in ten dimensions

Consider a $SU(5) \subset SO(10)$ structure specified by g, Ω, χ or equivalently by a chiral spinor η. As mentioned earlier, there always exists a connection ∇' that preserves the structure, $\nabla'\eta = 0$. In fact it is not unique and there is a whole family of such connections. The intrinsic torsion of the $SU(5)$ structure is the part of the torsion of an $SU(5)$ preserving connection that does not depend on the specific choice of such a connection. It can thus be thought of as an equivalence class of torsion tensors. Let us explain this in more detail.

Any metric preserving connection can be written as $\Gamma^r{}_{mn} = C^r{}_{mn} + K^r{}_{mn}$, where $C^r{}_{mn}$ are the Christoffel symbols and $K^r{}_{mn}$ is called the contorsion tensor. The contorsion satisfies the symmetry property $K_{rmn} = -K_{nmr}$ and the torsion can be determined by the contorsion by $T^r{}_{mn} = 2K^r{}_{[mn]}$. One can also construct the contorsion tensor from the torsion as

$$K^r{}_{mn} = \frac{1}{2}(T^r{}_{mn} + T_m{}^r{}_n + T_n{}^r{}_m) \tag{C.1}$$

Thus the torsion and contorsion are essentially equivalent.

The contorsion (and torsion) tensor lives in the space $T^* \otimes so(10) \simeq (T^* \otimes su(5)) \oplus (T^* \otimes su(5)^\perp)$, where $su(5)^\perp$ is the orthogonal complement of $su(5)$ in $so(10)$. The part of the contorsion tensor that lies in $T^* \otimes su(5)$ acts trivially on $SU(5)$ singlets such as Ω and χ. Thus any two connections that preserve the $SU(5)$ structure will differ by an element of $T^* \otimes su(5)$ and so the intrinsic contorsion, which we denote K^0, is the part of the contorsion that lies in $T^* \otimes su(5)^\perp$.

The space $T^* \otimes su(5)^\perp$, where the intrinsic contorsion lies, decomposes as

$$(\mathbf{5}+\bar{\mathbf{5}})\times(\mathbf{1}+\mathbf{10}+\bar{\mathbf{10}}) \to (\mathbf{10}+\bar{\mathbf{10}})+(\mathbf{40}+\bar{\mathbf{40}})+(\mathbf{45}+\bar{\mathbf{45}})+(\mathbf{5}+\bar{\mathbf{5}})+(\mathbf{5}'+\bar{\mathbf{5}}') \tag{C.2}$$

while the space $T^* \otimes su(5)$ decomposes as,

$$(\mathbf{5}+\bar{\mathbf{5}}) \times (\mathbf{24}) \to (\mathbf{5}+\bar{\mathbf{5}}) + (\mathbf{45}+\bar{\mathbf{45}}) + (\mathbf{70}+\bar{\mathbf{70}}) \tag{C.3}$$

We thus see that the contorsion tensor has, generically, three $\mathbf{5}+\bar{\mathbf{5}}$ and two $\mathbf{45}+\bar{\mathbf{45}}$ pieces. The most general contorsion tensor can thus be written

$$\begin{aligned} K_{rmn} &= T^{(1)}{}_m\Omega_{nr} + T^{(2)}{}_{[n}\Omega_{r]m} + T^{(3)}{}_{[n}g_{r]m} && \text{(C.4)} \\ &+ (P_0^{2,1}{}_{rmn} + \Omega_m{}^k Q_0^{2,1}{}_{knr} + c.c.) \\ &+ (S_0^{3,1}{}_{mk_1k_2k_3}\chi^{k_1k_2k_3}{}_{nr} + c.c.) \\ &+ (R^{3,0}_{mnr} + c.c.) + K^{70+\bar{70}}_{rmn} && \text{(C.5)} \end{aligned}$$

where $c.c.$ stands for complex conjugate, P, Q, R, S are forms of the type indicated. Demanding that the above connection preserves the $SU(5)$ structure allows one to relate the various components of the contorsion tensor to the W_i defined in (4.9).

To do this let ∇' be a covariant derivative with contorsion K that preserves the $SU(5)$ structure. To proceed write $\nabla'\Omega = \nabla'\chi = 0$ and then anti-symmetrise all of the indices to get

$$\begin{aligned} \frac{1}{6}d\Omega_{n_1n_2n_3} &= K^r{}_{[n_1n_2}\Omega_{|r|n_3]} \\ \frac{1}{30}d\chi_{n_1\ldots n_6} &= K^r{}_{[n_1n_2}\chi_{|r|n_3n_4n_5n_6]} && \text{(C.6)} \end{aligned}$$

It is now possible to explicitly relate the irreps of K to W_i by decomposing the left and right hand sides into $SU(5)$ irreps. We find that R and S are uniquely determined by W_1 and W_2, respectively, and that

$$\begin{aligned}(W_3)_{n_1n_2n_3} &= -6\Omega_{[n_1}{}^r P^{2,1}_0{}_{n_2n_3]r} - 2Q^{2,1}_0{}_{n_1n_2n_3} + c.c. \\ (W_4)_m &= -4\Omega_m{}^r T^{(2)}_r - 4T^{(3)}_m \\ (W_5)_m &= 40\Omega_m{}^r T^{(1)}_r + 12\Omega_m{}^r T^{(2)}_r + 20T^{(3)}\end{aligned} \tag{C.7}$$

The fact that the above components of the contorsion are not uniquely determined in terms of the W_i reflects the freedom in defining an $SU(5)$ preserving connection. Solving for $T^{(1)}, T^{(2)}$ in terms of $W_4, W_5, T^{(3)}$ and for $P^{2,1}_0$ in terms of $W_3, Q^{2,1}_0$ we conclude that the contorsion can be expressed as

$$\begin{aligned}K_{rmn} &= -\frac{1}{40}(\Omega W_5 + 3\Omega W_4)_m\Omega_{nr} + \frac{1}{4}(\Omega W_4)_{[n}\Omega_{r]m} + \frac{3}{2}\Omega_{[r}{}^k(W_3)_{mn]k} \\ &+ \frac{1}{4\cdot 4!}\chi_{rmn}{}^{k_1k_2}(W_1)_{k_1k_2} - \frac{1}{2\cdot 4!}\Omega_m{}^k(W_2)_{k\ell_1\ell_2\ell_3}\chi^{\ell_1\ell_2\ell_3}{}_{nr} \\ &+ \left(\frac{1}{5}\Omega T^{(3)}_m\Omega_{nr} + \Omega T^{(3)}{}_{[n}\Omega_{r]m} + T^{(3)}{}_{[n}g_{r]m}\right) \\ &+ \left(3\Omega_{[r}{}^k Q^{2,1}_0{}_{mn]k} + \Omega_m{}^k Q^{2,1}_0{}_{nrk} + c.c.\right) + K^{70+\bar{70}}_{rmn}\end{aligned} \tag{C.8}$$

where we have used the notation $\Omega V_m \equiv \Omega_m{}^r V_r$. The last three terms in the brackets act trivially on Ω and χ and so correspond to the terms appearing in the decomposition (C.3). Thus the intrinsic contorsion can be defined by (C.8) with the last three terms set to zero. We thus have

$$\begin{aligned}K^0_{rmn} &= -\frac{1}{40}(\Omega W_5 + 3\Omega W_4)_m\Omega_{nr} + \frac{1}{4}(\Omega W_4)_{[n}\Omega_{r]m} + \frac{3}{2}\Omega_{[r}{}^k(W_3)_{mn]k} \\ &+ \frac{1}{4\cdot 4!}\chi_{rmn}{}^{k_1k_2}(W_1)_{k_1k_2} - \frac{1}{2\cdot 4!}\Omega_m{}^k(W_2)_{k\ell_1\ell_2\ell_3}\chi^{\ell_1\ell_2\ell_3}{}_{nr}\end{aligned} \tag{C.9}$$

Equivalently we can calculate from this the intrinsic torsion thus showing that it is fully determined by the W_i, as claimed. For completeness we record the explicit form:

$$\begin{aligned}T^0_{rmn} &= \frac{1}{20}(2\Omega W_4 - \Omega W_5)_{[m}\Omega_{n]r} + \frac{1}{4}(\Omega W_4)_r\Omega_{mn} + 3\Omega_{[r}{}^k(W_3)_{mn]k} \\ &+ \frac{1}{2\cdot 4!}\chi_{rmn}{}^{k_1k_2}(W_1)_{k_1k_2} - \frac{1}{4!}\Omega_{[m}{}^k(W_2)_{|k\ell_1\ell_2\ell_3|}\chi^{\ell_1\ell_2\ell_3}{}_{n]r}\end{aligned} \tag{C.10}$$

Now since $\nabla^0 \equiv (\nabla + K^0)$ preserves the $SU(5)$ structure, where ∇ is the Levi-Civita connection, it leaves invariant the spinor η. Note that the spin connection of ∇^0 is related, in our conventions, to the spin connection of the Levi-Civita connection and the contorsion tensor by,

$$\omega^0{}_m{}^a{}_b = \omega_m{}^a{}_b + K^{0a}{}_{mb} \tag{C.11}$$

where $K^{0a}{}_{mb} \equiv e^a_r e^n_b K^{0r}{}_{mn}$. Using this and (C.9) we see that the spinor η solves

$$\begin{aligned}
[\nabla_m &+ \frac{1}{160}(\Omega W_5 + 5\Omega W_4)_m \Omega_{k_1k_2}\Gamma^{k_1k_2} - \frac{1}{16}(W_4)_k \Gamma_m{}^k \\
&+ \frac{1}{8}\Omega_m{}^r (W_3)_{rk_1k_2}\Gamma^{k_1k_2} - \frac{1}{394}\chi_{mk_1k_2}{}^{n_1n_2}(W_1)_{n_1n_2}\Gamma^{k_1k_2} \\
&+ \frac{1}{192}\Omega_m{}^r (W_2)_{r\ell_1\ell_2\ell_3}\chi^{\ell_1\ell_2\ell_3}{}_{k_1k_2}\Gamma^{k_1k_2}]\eta = 0
\end{aligned} \tag{C.12}$$

Let us make two further comments about $SU(5)$ structures that are not of direct relevance to the derivations in the text. Firstly, having got explicit expressions for the most general $SU(5)$ preserving connection we can easily see which $SU(5)$-structures admit a connection with totally antisymmetric torsion. From (C.8) we see that this requires that $T^{(3)}$ and $Q_0^{2,1}$ vanish and also that the structure must satisfy,

$$\begin{aligned}
W_2 &= 0 \\
8W_4 + W_5 &= 0
\end{aligned} \tag{C.13}$$

Secondly lets discuss how an $SU(5)$ structure is affected by a conformal transformation of the metric. Consider an $SU(5)$ structure (g, Ω, χ) and a transformation $g \to \tilde{g} = e^{2f} g$. The metric $\tilde{g}$ then admits an $SU(5)$ structure $(\tilde{g}, \tilde{\Omega}, \tilde{\chi})$ which is related to the original one by,

$$\begin{aligned}
\tilde{\Omega} &= e^{2f}\Omega \\
\tilde{\chi} &= e^{5f}\chi
\end{aligned} \tag{C.14}$$

The components W_i then transform as,

$$\begin{aligned}
\tilde{W}_1 &= e^f W_1 \\
\tilde{W}_2 &= e^{3f} W_2 \\
\tilde{W}_3 &= e^{2f} W_3 \\
\tilde{W}_4 &= W_4 + 8df \\
\tilde{W}_5 &= W_5 - 40df
\end{aligned} \tag{C.15}$$

We see that even though the components W_4 and W_5 transform nontrivially the combination $W_5 + 5W_4$ is conformally invariant. A trivial corollary is that a metric with an $SU(5)$ structure such that $W_5 = -5W_4$ and is exact is conformal to a Calabi-Yau fivefold.

D Some useful identities

We record here some of the identities satisfied by irreps of $SU(5)$ that are useful in deriving the results of table 1 and other formulae. Let $\Lambda^{(p,q)}$ denote a (p,q) form with a subscript of 0 denoting removal of traces, corresponding to an irreducible representation of $SU(5)$ (see (4.7)), then

$$
\begin{aligned}
*\Lambda_0^{(3,1)} &= -\Lambda_0^{(3,1)} \wedge \Omega \\
*\Lambda_0^{(2,2)} &= \Lambda_0^{(2,2)} \wedge \Omega \\
*\Lambda^{(2,0)} &= \frac{1}{3!}\Lambda^{(2,0)} \wedge \Omega^3 \\
*\Lambda_0^{(1,1)} &= -\frac{1}{3!}\Lambda_0^{(1,1)} \wedge \Omega^3 \\
*(\Lambda^{(2,0)} \wedge \Omega) &= \frac{1}{2}\Lambda^{(2,0)} \wedge \Omega^2 \\
*(\Lambda_0^{(1,1)} \wedge \Omega) &= -\frac{1}{2}\Lambda_0^{(1,1)} \wedge \Omega^2 \\
\Lambda_0^{(1,1)} \wedge \Omega^4 &= 0 \\
\Lambda_0^{(3,1)} \wedge \Omega^2 &= 0
\end{aligned}
\tag{D.1}
$$

The following identities are useful in deriving (4.28):

$$
\begin{aligned}
\Gamma_{a_1\dots.a_4}\epsilon &= -\frac{i}{2}\theta_{a_1\dots a_4 b}\Gamma^b\epsilon^* \\
\Gamma_{a_1\dots.a_3}\epsilon &= \frac{i}{2^2 2!}\theta_{a_1\dots a_3 b_1 b_2}\Gamma^{b_1 b_2}\epsilon^* \\
\Gamma_{a_1 a_2}\epsilon &= \frac{i}{2^3 3!}\theta_{a_1 a_2 b_1\dots b_3}\Gamma^{b_1\dots b_3}\epsilon^* \\
\Gamma_{a_1}\epsilon &= -\frac{i}{2^4 4!}\theta_{a_1 b_1\dots b_4}\Gamma^{b_1\dots b_4}\epsilon^* \\
\epsilon &= -\frac{i}{2^5 5!}\theta_{a_1\dots a_5}\Gamma^{a_1\dots a_5}\epsilon^*
\end{aligned}
\tag{D.2}
$$

E $SU(5)$ structures in D=11

In deriving the form of the most general geometry admitting a timelike Killing spinor we used the fact that K was a timelike Killing vector and then worked with the $SU(5)$ structure on the $D = 10$ base manifold, B, orthogonal to the orbits of K. Moreover, our analysis determined the type of $SU(5)$ structure on B.

Here we briefly indicate how our analysis also determines the type of D=11 $SU(5)$ structure specified by (K, Ω, Σ). For example, the fact that K is Killing expresses the vanishing of some components of the corresponding intrinsic torsion. One conceptual advantage of discussing the $SU(5)$ structure in $D = 11$ is that that the rotation parameter $d\omega$ arises as a component of the structure, while from the ten dimensional point of view it is just an arbitrary closed two form.

An $SU(5) \subset SO(10,1)$ structure in D=11 can be specified by a one form V, a two form J and a five form σ such that the vector dual to V is timelike, and the forms satisfy

$$\begin{aligned} i_V J &= 0 \\ i_V \sigma &= 0 \\ J \wedge \sigma &= 0 \\ J \wedge i_V * \sigma &= 0 \\ \sigma \wedge i_V * \sigma &= -2^4 \frac{J^5}{5!} \end{aligned} \tag{E.1}$$

The one form V allows us to reduce $SO(10,1) \to SO(10)$ and (J, σ) to further reduce $SO(10) \to SU(5)$. We require that the forms defining the structure have constant norm in the eleven dimensional metric (and so are related to rescaled versions of (K, Ω, Σ) as we shall see).

As we discussed in section 3, such structures are classified by the intrinsic torsion T^0 which lives in the space $T^* \otimes su(5)^\perp$ where $su(5) \oplus su(5)^\perp = so(10,1)$. The adjoint of $so(10,1)$ decomposes as $\mathbf{55} \to \mathbf{1} + (\mathbf{5} + \bar{\mathbf{5}}) + (\mathbf{10} + \bar{\mathbf{10}}) + \mathbf{24}$ and so the complement of $su(5)$ is given by $g^\perp = \mathbf{1} + (\mathbf{5} + \bar{\mathbf{5}}) + (\mathbf{10} + \bar{\mathbf{10}})$. Noting the following decomposition,

$$\begin{aligned} &(\mathbf{1} + \mathbf{5} + \bar{\mathbf{5}}) \otimes (\mathbf{1} + \mathbf{5} + \bar{\mathbf{5}} + \mathbf{10} + \bar{\mathbf{10}}) \to \\ &\mathbf{1} + \mathbf{1}' + \mathbf{1}'' + (\mathbf{5} + \bar{\mathbf{5}}) + (\mathbf{5} + \bar{\mathbf{5}})' + (\mathbf{5} + \bar{\mathbf{5}})'' + (\mathbf{10} + \bar{\mathbf{10}}) + (\mathbf{10} + \bar{\mathbf{10}})' + \\ &(\mathbf{10} + \bar{\mathbf{10}})'' + (\mathbf{15} + \bar{\mathbf{15}}) + \mathbf{24} + \mathbf{24}' + (\mathbf{40} + \bar{\mathbf{40}}) + (\mathbf{45} + \bar{\mathbf{45}}) \end{aligned} \tag{E.2}$$

we see that there are fourteen classes of $SU(5)$ structures in $D = 11$, and we can write

$$T^0 \in \bigoplus_{i=1}^{14} \mathcal{W}_i \tag{E.3}$$

The intrinsic torsion in each of the modules $\mathcal{W}_i$ can be expressed in terms of the exterior derivatives of (V, Ω, σ). To see this we decompose the exterior derivatives of the forms defining the structure and see which representation appear. Taking into account that they are $SU(5)$ invariant, we find

$$dV \quad \to \quad \mathbf{1} + \mathbf{24} + (\mathbf{10} + \bar{\mathbf{10}}) + (\mathbf{5} + \bar{\mathbf{5}})$$

$$
\begin{aligned}
dJ &\rightarrow (\mathbf{5}+\bar{\mathbf{5}})+(\mathbf{10}+\bar{\mathbf{10}})+(\mathbf{45}+\bar{\mathbf{45}})+\mathbf{1}+(\mathbf{10}+\bar{\mathbf{10}})'+\mathbf{24} \\
d\sigma &\rightarrow (\mathbf{5}+\bar{\mathbf{5}})+(\mathbf{10}+\bar{\mathbf{10}})+(\mathbf{40}+\bar{\mathbf{40}})+\mathbf{1}+\mathbf{1}'+(\mathbf{10}+\bar{\mathbf{10}})' \\
&+ (\mathbf{15}+\bar{\mathbf{15}})
\end{aligned} \tag{E.4}
$$

In comparing these to the irreps appearing in the intrinsic torsion, there appears to be a mismatch since four $\mathbf{1}$'s and five $\mathbf{10}+\bar{\mathbf{10}}$'s appear in (E.4) while in (E.2) we have only three $\mathbf{1}$'s and three $\mathbf{10}+\bar{\mathbf{10}}$'s. However this is not so since we can relate the $\mathbf{1}$ of dJ to the $\mathbf{1}+\mathbf{1}'$ of $d\sigma$ by the last condition in (E.1), while the $(\mathbf{10}+\bar{\mathbf{10}})$ of dJ and $d\sigma$ are related by $J\wedge\sigma=0$ and similarly for the $(\mathbf{10}+\bar{\mathbf{10}})'$.

As mentioned, the forms used above to define an $SU(5)$ structure in D=11 are not the ones constructed from the Killing spinors. They are related to those by,

$$
\begin{aligned}
K &= \Delta V \\
\Omega &= \Delta J \\
K\wedge\Sigma &= \Delta^2 V\wedge\sigma
\end{aligned} \tag{E.5}
$$

Given that we know expressions for $dK, d\Omega, d\chi$ we can obtain those for $dV, dJ, d\sigma$ and hence precisely determine the restrictions placed on the D=11 $\mathcal{W}_i$. In other words our analysis does indeed determine the D=11 $SU(5)$ structure as claimed.

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